

HOMOGENEOUS LORENTZ MANIFOLDS WITH SIMPLE ISOMETRY GROUP

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ABSTRACT. Let H be a closed, noncompact subgroup of a simple Lie group G , such that G/H admits an invariant Lorentz metric. We show that if $G = \mathrm{SO}(2, n)$, with $n \geq 3$, then the identity component H° of H is conjugate to $\mathrm{SO}(1, n)^\circ$. Also, if $G = \mathrm{SO}(1, n)$, with $n \geq 3$, then H° is conjugate to $\mathrm{SO}(1, n-1)^\circ$.

1. INTRODUCTION

- 1.1. Definition.** • A *Minkowski form* on a real vector space V is a nondegenerate quadratic form that is isometric to the form $-x_1^2 + x_2^2 + \cdots + x_{n+1}^2$ on \mathbb{R}^{n+1} , where $\dim V = n+1 \geq 2$.
- A *Lorentz metric* on a smooth manifold M is a choice of Minkowski metric on the tangent space $T_p M$, for each $p \in M$, such that the form varies smoothly as p varies.

A. Zeghib [Ze1] classified the compact homogeneous spaces that admit an invariant Lorentz metric. In this note, we remove the assumption of compactness, but add the restriction that the transitive group G is almost simple. Our starting point is a special case of a theorem of N. Kowalsky.

1.2. Theorem (N. Kowalsky, cf. [Ko3, Thm. 5.1]). *Let G/H be a nontrivial homogeneous space of a connected, almost simple Lie group G with finite center. If there is a G -invariant Lorentz metric on G/H , then either*

- 1) *there is also a G -invariant Riemannian metric on G/H ; or*
- 2) *G is locally isomorphic to either $\mathrm{SO}(1, n)$ or $\mathrm{SO}(2, n)$, for some n .*

As explained in the following elementary proposition, it is easy to characterize the homogeneous spaces that arise in Conclusion (1) of Theorem 1.2, although it is probably not reasonable to expect a complete classification.

1.3. Notation. We use \mathfrak{g} to denote the Lie algebra of a Lie group G , and $\mathfrak{h} \subset \mathfrak{g}$ to denote the Lie algebra of a Lie subgroup H of G .

1.4. Proposition (cf. [Ko3, Thm. 1.1]). *Let G/H be a homogeneous space of a Lie group G , such that \mathfrak{g} is simple and $\dim G/H \geq 2$. The following are equivalent.*

- 1) *The homogeneous space G/H admits both a G -invariant Riemannian metric and a G -invariant Lorentz metric.*
- 2) *The closure of $\mathrm{Ad}_G H$ is compact, and leaves invariant a one-dimensional subspace of \mathfrak{g} that is not contained in \mathfrak{h} .*

The two main results of this note examine the cases that arise in Conclusion (2) of Theorem 1.2. It is well known [Ko2, Egs. 2 and 3] that $\mathrm{SO}(1, n)^\circ / \mathrm{SO}(1, n - 1)^\circ$ and $\mathrm{SO}(2, n)^\circ / \mathrm{SO}(1, n)^\circ$ have invariant Lorentz metrics. Also, for any discrete subgroup Γ of $\mathrm{SO}(1, 2)$, the Killing form provides an invariant Lorentz metric on $\mathrm{SO}(1, 2)^\circ / \Gamma$. We show that these are essentially the only examples.

Note that $\mathrm{SO}(1, 1)$ and $\mathrm{SO}(2, 2)$ fail to be almost simple. Thus, in 1.2(2), we may assume

- G is locally isomorphic to $\mathrm{SO}(1, n)$, and $n \geq 2$; or
- G is locally isomorphic to $\mathrm{SO}(2, n)$, and $n \geq 3$.

2.3'. Proposition. *Let G be a Lie group that is locally isomorphic to $\mathrm{SO}(1, n)$, with $n \geq 2$.*

If H is a closed subgroup of G , such that

- *the closure of $\mathrm{Ad}_G H$ is not compact, and*
- *there is a G -invariant Lorentz metric on G/H ,*

then either

- 1) *after any identification of \mathfrak{g} with $\mathfrak{so}(1, n)$, the subalgebra \mathfrak{h} is conjugate to a standard copy of $\mathfrak{so}(1, n - 1)$ in $\mathfrak{so}(1, n)$, or*
- 2) *$n = 2$ and H is discrete.*

3.5'. Theorem. *Let G be a Lie group that is locally isomorphic to $\mathrm{SO}(2, n)$, with $n \geq 3$. If H is a closed subgroup of G , such that*

- *the closure of $\mathrm{Ad}_G H$ is not compact, and*
- *there is a G -invariant Lorentz metric on G/H ,*

then, after any identification of \mathfrak{g} with $\mathfrak{so}(2, n)$, the subalgebra \mathfrak{h} is conjugate to a standard copy of $\mathfrak{so}(1, n)$ in $\mathfrak{so}(2, n)$.

N. Kowalsky announced a much more general result than Theorem 3.5' in [Ko2, Thm. 4], but it seems that she did not publish a proof before her premature death. She announced a version of Proposition 2.3' (with much more general hypotheses and a somewhat weaker conclusion) in [Ko2, Thm. 3], and a proof appears in her Ph.D. thesis [Ko1, Cor. 6.2].

1.5. Remark. It is easy to see that there is a G -invariant Lorentz metric on G/H if and only if there is an $(\mathrm{Ad}_G H)$ -invariant Minkowski form on $\mathfrak{g}/\mathfrak{h}$. Thus, although Proposition 2.3' and Theorem 3.5' are geometric in nature, they can be restated in more algebraic terms. It is in such a form that they are proved in §2 and §3.

Proposition 2.3' and Theorem 3.5' are used in work of S. Adams [Ad3] on nontame actions on Lorentz manifolds. See [Zi, Ko3, AS, Ze2, Ad1, Ad2] for some other research concerning actions of Lie groups on Lorentz manifolds.

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2. HOMOGENEOUS SPACES OF $\mathrm{SO}(1, n)$

The following lemma is elementary.

2.1. Lemma. *Let π be the standard representation of $\mathfrak{g} = \mathfrak{so}(1, k)$ on \mathbb{R}^{k+1} , and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be an Iwasawa decomposition of \mathfrak{g} .*

- 1) *The representation π has only one positive weight (with respect to \mathfrak{a}), and the corresponding weight space is 1-dimensional.*
- 2) *There are subspaces V and W of \mathbb{R}^{k+1} , such that*
 - (a) $\dim(\mathbb{R}^{k+1}/V) = 1$;
 - (b) $\dim W = 1$;
 - (c) $\pi(\mathfrak{n})V \subset W$;
 - (d) *for all nonzero $u \in \mathfrak{n}$, we have $\pi(u)^2 \mathbb{R}^{k+1} = W$; and*
 - (e) *for all nonzero $u \in \mathfrak{n}$ and $v \in \mathbb{R}^{k+1}$, we have $\pi(u)^2 v = 0$ if and only if $v \in V$.*

2.2. Corollary. *Let \mathfrak{h} be a subalgebra of a real Lie algebra \mathfrak{g} , let Q be a Minkowski form on $\mathfrak{g}/\mathfrak{h}$, and define $\pi: N_G(\mathfrak{h}) \rightarrow \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$ by $\pi(g)(v + \mathfrak{h}) = (\mathrm{Ad}_G g)v + \mathfrak{h}$.*

- 1) *Suppose T is a connected Lie subgroup of G that normalizes H , such that $\pi(T) \subset \mathrm{SO}(Q)$ and $\mathrm{Ad}_G T$ is diagonalizable over \mathbb{R} . Then, for any ordering of the T -weights on \mathfrak{g} , the subalgebra \mathfrak{h} contains codimension-one subspaces of both \mathfrak{g}^+ and \mathfrak{g}^- , where \mathfrak{g}^+ is the sum of all the positive weight spaces of T , and \mathfrak{g}^- is the sum of all the negative weight spaces of T .*
- 2) *If U is a connected Lie subgroup of G that normalizes H , such that $\pi(U) \subset \mathrm{SO}(Q)$ and $\mathrm{Ad}_G U$ is unipotent, then there are subspaces V/\mathfrak{h} and W/\mathfrak{h} of $\mathfrak{g}/\mathfrak{h}$, such that*
 - (a) $\dim(\mathfrak{g}/V) = 1$;
 - (b) $\dim(W/\mathfrak{h}) = 1$;
 - (c) $[V, \mathfrak{u}] \subset W$;
 - (d) *for each $u \in \mathfrak{u}$, either $W = \mathfrak{h} + (\mathrm{ad}_{\mathfrak{g}} u)^2 \mathfrak{g}$, or $[\mathfrak{g}, u] \subset \mathfrak{h}$; and*
 - (e) *for all $u \in \mathfrak{u}$, we have $(\mathrm{ad}_{\mathfrak{g}} u)^2 V \subset \mathfrak{h}$.*

2.3. Proposition. *Let H be a Lie subgroup of $G = \mathrm{SO}(1, n)$, with $n \geq 2$, such that*

- *the closure of H is not compact; and*
- *there is an $(\mathrm{Ad}_G H)$ -invariant Minkowski form on $\mathfrak{g}/\mathfrak{h}$.*

Then either

- 1) *H° is conjugate to a standard copy of $\mathrm{SO}(1, n-1)^\circ$ in $\mathrm{SO}(1, n)$, or*
- 2) *$n = 2$ and H° is trivial.*

Proof. Let \overline{H} be the Zariski closure of H , and note that the Minkowski form is also invariant under $\mathrm{Ad}_G \overline{H}$. Replacing H by a finite-index subgroup, we may assume \overline{H} is Zariski connected.

Let $G = KAN$ be an Iwasawa decomposition of G .

Case 1. *Assume $n \geq 3$ and $A \subset \overline{H}$. From Corollary 2.2(1), we see that \mathfrak{h} contains codimension-one subspaces of both \mathfrak{n} and \mathfrak{n}^- . (Note that this implies H° is nontrivial.) This implies that \overline{H} is reductive. (Because $(H \cap N)^\circ \mathrm{unip} \overline{H}$ is a unipotent subgroup that intersects N nontrivially (and $\mathbb{R}\text{-rank } G = 1$), it must be contained in N , so $\mathrm{unip} \overline{H} \subset N$. Similarly, $\mathrm{unip} \overline{H} \subset N^-$. Therefore $\mathrm{unip} \overline{H} \subset N \cap N^- = e$.) Then, since \overline{H} contains a codimension-one subgroup of N , and since $A \subset \overline{H}$, it follows that \overline{H} is conjugate to either $\mathrm{SO}(1, n-1)$ or $\mathrm{SO}(1, n)$. Because H° is a nontrivial, connected, normal subgroup of \overline{H} , we conclude that H° is conjugate to either $\mathrm{SO}(1, n-1)^\circ$ or $\mathrm{SO}(1, n)^\circ$. Because $\mathfrak{g}/\mathfrak{h} \neq 0$ (else*

$\dim \mathfrak{g}/\mathfrak{h} = 0 < 2$, which contradicts the fact that there is a Minkowski form on $\mathfrak{g}/\mathfrak{h}$, we see that H° is conjugate to $\mathrm{SO}(1, n-1)^\circ$.

Case 2. Assume $n \geq 3$ and \overline{H} does not contain any nontrivial hyperbolic elements. The Levi subgroup of \overline{H} must be compact, and the radical of \overline{H} must be unipotent, so choose a compact M and a nontrivial unipotent subgroup U such that $\overline{H} = M \ltimes U$. Replacing H by a conjugate, we may assume, without loss of generality, that $U \subset N$.

Let us show, for every nonzero $u \in \mathfrak{u}$, that $[\mathfrak{g}, u] \not\subset \mathfrak{h}$. From the Morosov Lemma [Ja, Thm. 17(1), p. 100], we know there exists $v \in \mathfrak{g}$, such that $[v, u]$ is hyperbolic (and nonzero). If $[v, u] \in \mathfrak{h}$, this contradicts the fact that \overline{H} does not contain nontrivial hyperbolic elements.

Let V/\mathfrak{h} and W/\mathfrak{h} be subspaces of $\mathfrak{g}/\mathfrak{h}$ as in Corollary 2.2(2). Because $(\mathrm{ad}_{\mathfrak{g}} u)^2 \mathfrak{g} = \mathfrak{n}$ for every nonzero $u \in \mathfrak{n}$, we have $W = \mathfrak{n} + \mathfrak{h}$ (see 2.2(2d)), so $\dim \mathfrak{n}/(\mathfrak{h} \cap \mathfrak{n}) = 1$ (see 2.2(2b)) and

$$(2.4) \quad [\mathfrak{u}, V] \subset W = \mathfrak{n} + \mathfrak{h} \subset \mathfrak{n} + \overline{\mathfrak{h}} = \mathfrak{n} + \mathfrak{m}$$

(see 2.2(2c)).

Assume, for the moment, that $n \geq 4$. Then

$$\begin{aligned} \dim \mathfrak{u} + \dim(V \cap \mathfrak{n}^-) &\geq \dim(\mathfrak{h} \cap \mathfrak{n}) + \dim(V \cap \mathfrak{n}^-) \geq (\dim \mathfrak{n} - 1) + (\dim \mathfrak{n}^- - 1) \\ &= (n - 2) + (n - 2) \geq n > \dim \mathfrak{n}. \end{aligned}$$

This implies that there exist $u \in \mathfrak{u}$ and $v \in V \cap \mathfrak{n}^-$, such that $\langle u, v \rangle \cong \mathfrak{sl}(2, \mathbb{R})$, with $[u, v]$ hyperbolic (and nonzero). This contradicts the fact that $\mathfrak{m} + \mathfrak{n}$ has no nontrivial hyperbolic elements.

We may now assume that $n = 3$. For any nonzero $u \in \mathfrak{n}$, we have

$$\dim[u, V] \geq \dim[u, \mathfrak{g}] - 1 = \dim \mathfrak{n} + 1 > \dim \mathfrak{n},$$

so $[u, V] \not\subset \mathfrak{n}$. Then, from (2.4), we conclude that $\mathfrak{m} \neq 0$, so \mathfrak{m} acts irreducibly on \mathfrak{n} . This contradicts the fact that $\mathfrak{h} \cap \mathfrak{n}$ is a codimension-one subspace of \mathfrak{n} that is normalized by \mathfrak{m} .

Case 3. Assume $n = 2$. We may assume H° is nontrivial (otherwise Conclusion (2) holds). We must have $\dim \mathfrak{g}/\mathfrak{h} \geq 2$, so we conclude that $\dim H^\circ = 1$ and $\dim \mathfrak{g}/\mathfrak{h} = 2$. Because $\mathrm{SO}(1, 1)$ consists of hyperbolic elements, this implies that $\mathrm{Ad}_G h$ acts diagonalizably on $\mathfrak{g}/\mathfrak{h}$, for every $h \in H$. Therefore H° is conjugate to A , and, hence, to $\mathrm{SO}(1, 1)^\circ$. \square

3. HOMOGENEOUS SPACES OF $\mathrm{SO}(2, n)$

3.1. Theorem (Borel-Tits [BT2, Prop. 3.1]). *Let H be an F -subgroup of a reductive algebraic group G over a field F of characteristic zero. Then there is a parabolic F -subgroup P of G , such that $\mathrm{unip} H \subset \mathrm{unip} P$ and $H \subset N_G(\mathrm{unip} H) \subset P$.*

3.2. Notation. Let $k = \lfloor n/2 \rfloor$. Identifying \mathbb{C}^{k+1} with \mathbb{R}^{2k+2} yields an embedding of $\mathrm{SU}(1, k)$ in $\mathrm{SO}(2, 2k)$. Then the inclusion $\mathbb{R}^{2k+2} \hookrightarrow \mathbb{R}^{2k+3}$ yields an embedding of $\mathrm{SU}(1, k)$ in $\mathrm{SO}(2, 2k+1)$. Thus, we may identify $\mathrm{SU}(1, \lfloor n/2 \rfloor)$ with a subgroup of $\mathrm{SO}(2, n)$.

We use the following well-known result to shorten one case of the proof of Theorem 3.5.

3.3. Lemma ([OW, Lem. 6.8]). *If L is a connected, almost-simple subgroup of $\mathrm{SO}(2, n)$, such that \mathbb{R} -rank $L = 1$ and $\dim L > 3$, then L is conjugate under $\mathrm{O}(2, n)$ to a subgroup of either $\mathrm{SO}(1, n)$ or $\mathrm{SU}(1, \lfloor n/2 \rfloor)$.*

3.4. Corollary. *Let L be a connected, reductive subgroup of $G = \mathrm{SO}(2, n)$, such that $\mathbb{R}\text{-rank } L = 1$. Then $\dim U \leq n - 1$, for every connected, unipotent subgroup U of L .*

Furthermore, if $\dim U = n - 1$, then either

- 1) *L is conjugate to $\mathrm{SO}(1, n)^\circ$; or*
- 2) *n is even, and L is conjugate under $\mathrm{O}(2, n)$ to $\mathrm{SU}(1, n/2)$.*

3.5. Theorem. *Let H be a Lie subgroup of $G = \mathrm{SO}(2, n)$, with $n \geq 3$, such that*

- *the closure of H is not compact, and*
- *there is an $(\mathrm{Ad}_G H)$ -invariant Minkowski form on $\mathfrak{g}/\mathfrak{h}$.*

Then H° is conjugate to a standard copy of $\mathrm{SO}(1, n)^\circ$ in $\mathrm{SO}(2, n)$.

Proof. Let \overline{H} be the Zariski closure of H , and note that the Minkowski form is also invariant under $\mathrm{Ad}_G \overline{H}$. Replacing H by a finite-index subgroup, we may assume \overline{H} is Zariski connected.

Let $G = KAN$ be an Iwasawa decomposition of G . For each real root ϕ of \mathfrak{g} (with respect to the Cartan subalgebra \mathfrak{a}), let \mathfrak{g}_ϕ be the corresponding root space, and let $\mathrm{proj}_\phi: \mathfrak{g} \rightarrow \mathfrak{g}_\phi$ and $\mathrm{proj}_{\phi \oplus -\phi}: \mathfrak{g} \rightarrow \mathfrak{g}_\phi + \mathfrak{g}_{-\phi}$ be the natural projections. Fix a choice of simple real roots α and β of \mathfrak{g} , such that $\dim \mathfrak{g}_\alpha = 1$ and $\dim \mathfrak{g}_\beta = n - 2$ (so the positive real roots are α , β , $\alpha + \beta$, and $\alpha + 2\beta$). Replacing N by a conjugate under the Weyl group, we may assume $\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}$. From the classification of parabolic subgroups [BT1, Prop. 5.14, p. 99], we know that the only proper parabolic subalgebras of \mathfrak{g} that contain $\mathfrak{n}_\mathfrak{g}(\mathfrak{n})$ are

$$(3.6) \quad \mathfrak{n}_\mathfrak{g}(\mathfrak{n}), \quad \mathfrak{p}_\alpha = \mathfrak{n}_\mathfrak{g}(\mathfrak{n}) + \mathfrak{g}_{-\alpha}, \quad \text{and} \quad \mathfrak{p}_\beta = \mathfrak{n}_\mathfrak{g}(\mathfrak{n}) + \mathfrak{g}_{-\beta}.$$

Case 1. *Assume $\overline{\mathfrak{h}}$ contains nontrivial hyperbolic elements.* Let $\mathfrak{t} = \overline{\mathfrak{h}} \cap \mathfrak{a}$. Replacing H by a conjugate, we may assume $\mathfrak{t} \neq 0$.

Subcase 1.1. *Assume $\mathfrak{t} \in \{\ker(\alpha + \beta), \ker \beta\}$.*

Subsubcase 1.1.1. *Assume \overline{H} is reductive.* We may assume $\mathfrak{t} = \ker(\alpha + \beta)$ (if necessary, replace H with its conjugate under the Weyl reflection corresponding to the root α). Then, from Corollary 2.2(1), we see that \mathfrak{h} contains a codimension-one subspace of $\mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta + \mathfrak{g}_{-\alpha}$. (Note that this implies H° is nontrivial.)

Let $\mathfrak{n}' = \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta + \mathfrak{g}_{-\alpha}$, so \mathfrak{n}' is the Lie algebra of a maximal unipotent subgroup of G . (In fact, \mathfrak{n}' is the image of \mathfrak{n} under the Weyl reflection corresponding to the root α .) From the preceding paragraph, we know that

$$\dim(\overline{\mathfrak{h}} \cap \mathfrak{n}') \geq \dim(\mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta + \mathfrak{g}_{-\alpha}) - 1 = n - 1.$$

Therefore, Corollary 3.4 implies that \overline{H} is conjugate (under $\mathrm{O}(2, n)$) to either $\mathrm{SO}(1, n)$ or $\mathrm{SU}(1, n/2)$. It is easy to see that \overline{H} is not conjugate to $\mathrm{SU}(1, n/2)$. (See [OW, proof of Thm. 1.5] for an explicit description of $\mathfrak{su}(1, n/2) \cap \mathfrak{n}$. If n is even, then $n > 3$, so $\mathfrak{su}(1, n/2)$ does not contain a codimension-one subspace of any $(n - 2)$ -dimensional root space, but $\overline{\mathfrak{h}}$ does contain a codimension-one subspace of \mathfrak{g}_β .) Therefore, we conclude that \overline{H} is conjugate to $\mathrm{SO}(1, n)$. Then, because H° is a nontrivial, connected, normal subgroup of \overline{H} , we conclude that $H^\circ = (\overline{H})^\circ$ is conjugate to $\mathrm{SO}(1, n)^\circ$.

Subsubcase 1.1.2. *Assume \overline{H} is not reductive.* Let P be a maximal parabolic subgroup of G that contains \overline{H} (see Theorem 3.1). By replacing P and H with conjugate subgroups,

we may assume that P contains the minimal parabolic subgroup $N_G(N)$. Therefore, the classification of parabolic subalgebras (3.6) implies that P is either P_α or P_β .

Subsubsubcase 1.1.2.1. Assume $\mathfrak{t} = \ker(\alpha + \beta)$. From Corollary 2.2(1), we see that \mathfrak{h} (and hence also \mathfrak{p}) contains codimension-one subspaces of $\mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta + \mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-\alpha-2\beta} + \mathfrak{g}_{-\beta} + \mathfrak{g}_\alpha$. Because \mathfrak{p}_α does not contain such a subspace of $\mathfrak{g}_{-\alpha-2\beta} + \mathfrak{g}_{-\beta} + \mathfrak{g}_\alpha$, we conclude that $P = P_\beta$. Furthermore, because the intersection of \mathfrak{p}_β with each of these subspaces does have codimension one, we conclude that \mathfrak{h} has precisely the same intersection; therefore $(\mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta) + (\mathfrak{g}_{-\beta} + \mathfrak{g}_\alpha) \subset \mathfrak{h}$. Hence $\mathfrak{h} \supset [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. We now have

$$(\text{ad}_{\mathfrak{g}} \mathfrak{g}_{\alpha+\beta})^2 \mathfrak{g} = \mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta} \equiv 0 \pmod{\mathfrak{h}},$$

so Corollary 2.2(2d) implies

$$\mathfrak{h} \supset [\mathfrak{g}, \mathfrak{g}_{\alpha+\beta}] \supset [\mathfrak{g}_{-\alpha-\beta}, \mathfrak{g}_{\alpha+\beta}] \supset \ker \beta.$$

This contradicts the fact that $\overline{\mathfrak{h}} \cap \mathfrak{a} = \mathfrak{t} = \ker(\alpha + \beta)$.

Subsubsubcase 1.1.2.2. Assume $\mathfrak{t} = \ker \beta$. From Corollary 2.2(1), we see that \mathfrak{h} (and hence also \mathfrak{p}) contains a codimension-one subspace of $\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-\alpha-\beta} + \mathfrak{g}_{-\alpha-2\beta}$. Because neither \mathfrak{p}_α nor \mathfrak{p}_β contains such a subspace, this is a contradiction.

Subcase 1.2. Assume $\mathfrak{t} \in \{\ker \alpha, \ker(\alpha + 2\beta)\}$. We may assume $\mathfrak{t} = \ker \alpha$ (if necessary, replace H with its conjugate under the Weyl reflection corresponding to the root β). From Corollary 2.2(1), we see that \mathfrak{h} contains a codimension-one subspace of $\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}$. Because any codimension-one subalgebra of a nilpotent Lie algebra must contain the commutator subalgebra, we conclude that \mathfrak{h} contains $\mathfrak{g}_{\alpha+2\beta}$. Then we have

$$(\text{ad}_{\mathfrak{g}} \mathfrak{g}_{\alpha+2\beta})^2 \mathfrak{g} = \mathfrak{g}_{\alpha+2\beta} \equiv 0 \pmod{\mathfrak{h}},$$

so Corollary 2.2(2d) implies

$$\mathfrak{h} \supset [\mathfrak{g}, \mathfrak{g}_{\alpha+2\beta}] \supset \mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}.$$

Similarly, we also have $\mathfrak{h} \supset \mathfrak{g}_{-\beta} + \mathfrak{g}_{-\alpha-\beta} + \mathfrak{g}_{-\alpha-2\beta}$. It is now easy to show that $\mathfrak{h} \supset \mathfrak{g}_\phi$ for every real root ϕ , so $\mathfrak{h} = \mathfrak{g}$. This contradicts the fact that $\mathfrak{g}/\mathfrak{h} \neq 0$.

Subcase 1.3. Assume \mathfrak{t} contains a regular element of \mathfrak{a} . Replacing H by a conjugate under the Weyl group, we may assume that \mathfrak{n} is the sum of the positive root spaces, with respect to \mathfrak{t} . Then, from Corollary 2.2(1), we see that \mathfrak{h} contains codimension-one subspaces of both \mathfrak{n} and \mathfrak{n}^- . Therefore, \mathfrak{h} contains codimension-one subspaces of $\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}$ and $\mathfrak{g}_{-\beta} + \mathfrak{g}_{-\alpha-\beta} + \mathfrak{g}_{-\alpha-2\beta}$, so the argument of Subcase 1.2 applies.

Case 2. Assume $\overline{\mathfrak{h}}$ does not contain nontrivial hyperbolic elements. The Levi subgroup of \overline{H} must be compact, and the radical of \overline{H} must be unipotent, so choose a compact M and a nontrivial unipotent subgroup U such that $\overline{H} = M \ltimes U$. Choose subspaces V/\mathfrak{h} and W/\mathfrak{h} of $\mathfrak{g}/\mathfrak{h}$ as in Corollary 2.2(2).

Let P be a proper parabolic subgroup of G , such that $U \subset \text{unip } P$ and $H \subset P$ (see Theorem 3.1). Replacing H and P by conjugates, we may assume, without loss of generality, that P contains the minimal parabolic subgroup $N_G(N)$ (so $\text{unip } P \subset N$). From the classification of parabolic subalgebras (3.6), we know that there are only three possibilities for P . We consider each of these possibilities separately.

First, though, let us show that

$$(3.7) \quad \text{for every nonzero } u \in \mathfrak{u}, \text{ we have } [\mathfrak{g}, u] \not\subset \mathfrak{h}.$$

From the Morosov Lemma [Ja, Thm. 17(1), p. 100], we know there exists $v \in \mathfrak{g}$, such that $[v, u]$ is hyperbolic (and nonzero). If $[v, u] \in \mathfrak{h}$, this contradicts the fact that $\bar{\mathfrak{h}}$ does not contain nontrivial hyperbolic elements.

Subcase 2.1. Assume $P = N_G(N)$ is a minimal parabolic subgroup of G .

Subsubcase 2.1.1. Assume $\text{proj}_\beta \mathfrak{u} \neq 0$. Choose $u \in \mathfrak{u}$, such that $\text{proj}_\beta u \neq 0$, and let $Z = (\text{ad}_{\mathfrak{g}} u)^2 \mathfrak{g}_{-\alpha-2\beta}$. (So $\dim Z = 1$, $\text{proj}_{-\alpha} Z \neq 0$, and $\text{proj}_{-\alpha-\beta} Z = 0$.) From Corollary 2.2(2d), we know that $Z \subset W$. Then, because $\text{proj}_{-\alpha} \mathfrak{h} \subset \text{proj}_{-\alpha} \mathfrak{p} = 0$, we conclude, from Corollary 2.2(2b), that $W = \mathfrak{h} + Z$.

Because $W = \mathfrak{h} + Z \subset \mathfrak{p} + Z$, we have $\text{proj}_{-\alpha-\beta} W = 0$. Therefore, because $\text{proj}_\beta u \neq 0$, we conclude, from Corollary 2.2(2c), that $\text{proj}_{-\alpha-2\beta} V = 0$, so Corollary 2.2(2a) implies that $V = \ker(\text{proj}_{-\alpha-2\beta})$. In particular, we have $\mathfrak{g}_{-\beta} \subset V$, so Corollary 2.2(2c) implies $[\mathfrak{g}_{-\beta}, u] \subset W$. Therefore, we have

$$\begin{aligned} [\mathfrak{g}_{-\beta}, \text{proj}_\beta u] &\subset [\mathfrak{g}_{-\beta}, u + (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta})] = [\mathfrak{g}_{-\beta}, u] + [\mathfrak{g}_{-\beta}, \mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}] \\ &\subset W + (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta}) = \mathfrak{h} + Z + (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta}) \subset \mathfrak{m} + \mathfrak{n} + Z. \end{aligned}$$

Because $\text{proj}_{-\alpha} [\mathfrak{g}_{-\beta}, \text{proj}_\beta u] = 0$, we conclude that $[\mathfrak{g}_{-\beta}, \text{proj}_\beta u] \subset \mathfrak{m} + \mathfrak{n}$. This contradicts the fact that $\mathfrak{m} + \mathfrak{n}$ does not contain nontrivial hyperbolic elements.

Subsubcase 2.1.2. Assume $\text{proj}_\beta \mathfrak{u} = 0$. Replacing H by a conjugate under N , we may assume $\mathfrak{m} \subset \mathfrak{g}_0$, so $\text{proj}_\beta \bar{\mathfrak{h}} = 0$.

We have $\mathfrak{u} \subset \mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}$, so $(\text{ad}_{\mathfrak{g}} u)^2 \mathfrak{g} \subset \mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}$ for every $u \in \mathfrak{u}$. Thus, Corollary 2.2(2d) implies $W \subset (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}) + \mathfrak{h}$.

We have

$$\text{proj}_{\beta \oplus -\beta} W \subset \text{proj}_{\beta \oplus -\beta} (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}) + \text{proj}_{\beta \oplus -\beta} \mathfrak{h} = 0,$$

so Corollary 2.2(2c) implies that $\text{proj}_{\beta \oplus -\beta} ((\text{ad}_{\mathfrak{g}} u)V) = 0$.

Subsubsubcase 2.1.2.1. Assume $\text{proj}_\alpha u \neq 0$, for some $u \in \mathfrak{u}$. From the conclusion of the preceding paragraph, we know that $\text{proj}_{-\beta} ((\text{ad}_{\mathfrak{g}} u)V) = 0$. Because $\text{proj}_\beta u = 0$ and $\text{proj}_\alpha \neq 0$, this implies $\text{proj}_{-\alpha-\beta} V = 0$, so $V = \ker(\text{proj}_{-\alpha-\beta})$ (see 2.2(2a)). In particular, $\mathfrak{g}_{-\alpha} \subset V$, so Corollary 2.2(2c) implies

$$\begin{aligned} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] &\subset [\mathfrak{u} + (\mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}), \mathfrak{g}_{-\alpha}] \subset [\mathfrak{u}, V] + [\mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}, \mathfrak{g}_{-\alpha}] \\ &\subset W + \mathfrak{g}_\beta \subset \mathfrak{h} + \mathfrak{n} \subset \mathfrak{m} + \mathfrak{n}. \end{aligned}$$

This contradicts the fact that $\mathfrak{m} + \mathfrak{n}$ does not contain nontrivial hyperbolic elements.

Subsubsubcase 2.1.2.2. Assume $\text{proj}_{\alpha+\beta} u \neq 0$, for some $u \in \mathfrak{u}$. From Subsubsubcase 2.1.2.1, we may assume $\text{proj}_\alpha u = 0$. Because $0 = \text{proj}_{\beta \oplus -\beta} ((\text{ad}_{\mathfrak{g}} u)V)$ has codimension ≤ 1 in $\text{proj}_{\beta \oplus -\beta} ((\text{ad}_{\mathfrak{g}} u)\mathfrak{g})$ (see 2.2(2a)), which contains the 2-dimensional subspace $\text{proj}_{\beta \oplus -\beta} ([u, \mathfrak{g}_{-\alpha-2\beta} + \mathfrak{g}_{-\alpha}])$, we have a contradiction.

Subsubsubcase 2.1.2.3. Assume $\mathfrak{u} = \mathfrak{g}_{\alpha+2\beta}$. (This argument is similar to Subsubsubcase 2.1.2.1.) Because $\text{proj}_\beta ((\text{ad}_{\mathfrak{g}} u)V) = 0$, we know that $\text{proj}_{-\alpha-\beta} V = 0$, so $V = \ker(\text{proj}_{-\alpha-\beta})$ (see 2.2(2a)). In particular, $\mathfrak{g}_{-\alpha-2\beta} \subset V$, so Corollary 2.2(2c) implies

$$[\mathfrak{g}_{\alpha+2\beta}, \mathfrak{g}_{-\alpha-2\beta}] \subset [\mathfrak{u}, V] \subset W \subset \mathfrak{h} + \mathfrak{n} \subset \mathfrak{m} + \mathfrak{n}.$$

This contradicts the fact that $\mathfrak{m} + \mathfrak{n}$ does not contain nontrivial hyperbolic elements.

Subcase 2.2. Assume $P = P_\alpha$. We may assume there exists $x \in \mathfrak{h}$, such that $\text{proj}_{-\alpha} x \neq 0$ (otherwise, $H \subset N_G(N)$, so Subcase 2.1 applies). Note that, because $U \subset \text{unip } P$, we have $\text{proj}_\alpha \mathfrak{u} = 0$.

Subsubcase 2.2.1. Assume $\text{proj}_{\alpha+\beta} \mathfrak{u} \neq 0$. Choose $u \in \mathfrak{u}$, such that $\text{proj}_{\alpha+\beta} u \neq 0$. Then $[x, u] \in [\mathfrak{h}, \mathfrak{u}] \subset \mathfrak{u}$, and $[[x, u], u]$ is a nonzero element of $\mathfrak{g}_{\alpha+2\beta}$, so we see that $\mathfrak{g}_{\alpha+2\beta} \subset [\mathfrak{u}, \mathfrak{u}]$. Because every unipotent subgroup of $\text{SO}(1, k)$ is abelian, we conclude that $\text{ad}_{\mathfrak{g}} \mathfrak{g}_{\alpha+2\beta}$ acts trivially on $\mathfrak{g}/\mathfrak{h}$, which means $\mathfrak{h} \supset [\mathfrak{g}, \mathfrak{g}_{\alpha+2\beta}]$. This contradicts (3.7).

Subsubcase 2.2.2. Assume $\text{proj}_{\alpha+\beta} \mathfrak{u} = 0$. We may assume, furthermore, that $\text{proj}_\alpha \mathfrak{h} \neq 0$ (otherwise, by replacing H with its conjugate under the Weyl reflection corresponding to the root α , we could revert to Subcase 2.1). Then, because $[\mathfrak{h}, \mathfrak{u}] \subset \mathfrak{u}$, we must have $\text{proj}_\beta \mathfrak{u} = 0$. Thus, $\mathfrak{u} = \mathfrak{g}_{\alpha+2\beta}$. From Corollary 2.2(2d), we have

$$W = [\mathfrak{g}, \mathfrak{g}_{\alpha+2\beta}, \mathfrak{g}_{\alpha+2\beta}] + \mathfrak{h} = \mathfrak{g}_{\alpha+2\beta} + \mathfrak{h} \subset \mathfrak{u} + \bar{\mathfrak{h}} = \bar{\mathfrak{h}},$$

so

$$\begin{aligned} W \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta}) &\subset \bar{\mathfrak{h}} \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta}) = (\bar{\mathfrak{h}} \cap \mathfrak{n}) \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta}) \\ &= \mathfrak{u} \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta}) = \mathfrak{g}_{\alpha+2\beta} \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta}) = 0. \end{aligned}$$

On the other hand, from Corollary 2.2(2c), we know that W contains a codimension-one subspace of $[\mathfrak{g}, \mathfrak{g}_{\alpha+2\beta}]$, so W contains a codimension-one subspace of $\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta}$. This is a contradiction.

Subcase 2.3. Assume $P = P_\beta$. Note that, because $U \subset \text{unip } P$, we have $\text{proj}_\beta \mathfrak{u} = 0$.

From Corollary 2.2(2d), we have

$$\begin{aligned} W &= \mathfrak{h} + (\text{ad}_{\mathfrak{g}} u)^2 \mathfrak{g} \subset \mathfrak{h} + (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}) \\ &= \mathfrak{h} + \text{unip } \mathfrak{p}_\beta \subset (\mathfrak{m} + \mathfrak{u}) + \text{unip } \mathfrak{p}_\beta = \mathfrak{m} + \text{unip } \mathfrak{p}_\beta. \end{aligned}$$

Subsubcase 2.3.1. Assume there is some nonzero $u \in \mathfrak{u}$, such that $\text{proj}_\alpha u = 0$. Replacing H by a conjugate (under $G_{-\beta}$), we may assume $\text{proj}_{\alpha+\beta} u \neq 0$.

Let $V' = V \cap (\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-\alpha-\beta})$. Because V' contains a codimension-one subspace of $\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-\alpha-\beta}$ (see Corollary 2.2(2a)), one of the following two subsubsubcases must apply.

Subsubsubcase 2.3.1.1. Assume there exists $v \in V'$, such that $\text{proj}_{-\alpha-\beta} v = 0$. From Corollary 2.2(2c), we have $[u, v] \in W$. Then, because $[u, v]$ is a nonzero element of \mathfrak{g}_β , we conclude that

$$0 \neq W \cap \mathfrak{g}_\beta \subset (\mathfrak{m} + \text{unip } \mathfrak{p}_\beta) \cap \mathfrak{g}_\beta = 0.$$

This contradicts the fact that M , being compact, has no nontrivial unipotent elements.

Subsubsubcase 2.3.1.2. Assume $\text{proj}_{-\alpha-\beta} V' = \mathfrak{g}_{-\alpha-\beta}$. For $v \in V'$, we have $\text{proj}_0[u, v] = [\text{proj}_{\alpha+\beta} u, \text{proj}_{-\alpha-\beta} v]$. Thus, there is some $v \in V'$, such that $\text{proj}_0[u, v]$ is hyperbolic (and nonzero). On the other hand, from Corollary 2.2(2c), we have $[u, v] \in W = \mathfrak{m} + \text{unip } \mathfrak{p}_\beta$. This contradicts the fact that $\mathfrak{m} \subset \bar{\mathfrak{h}}$ does not contain nonzero hyperbolic elements.

Subsubcase 2.3.2. Assume $\text{proj}_\alpha u \neq 0$, for every nonzero $u \in \mathfrak{u}$. Fix some nonzero $u \in \mathfrak{u}$. Because $\dim \mathfrak{u}_\alpha = 1$, we must have $\dim \mathfrak{u} = 1$ (so $\mathfrak{u} = \mathbb{R}u$). Replacing H by a conjugate (under G_β), we may assume $\text{proj}_{\alpha+\beta} u = 0$. Also, we may assume $\text{proj}_{\alpha+2\beta} u \neq 0$ (otherwise, we could revert to Subsubcase 2.3.1 by replacing H with its conjugate under the Weyl reflection corresponding to the root β).

Let $\mathfrak{t} = [u, \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-\alpha-2\beta}]$. Because $\langle \mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha} \rangle$ and $\langle \mathfrak{g}_{\alpha+2\beta}, \mathfrak{g}_{-\alpha-2\beta} \rangle$ centralize each other, we see that $\mathfrak{t} = [\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}] + [\mathfrak{g}_{\alpha+2\beta}, \mathfrak{g}_{-\alpha-2\beta}]$ is a two-dimensional subspace of \mathfrak{g} consisting entirely of hyperbolic elements. Because V contains a codimension-one subspace of $\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-\alpha-2\beta}$ (see Corollary 2.2(2a)), and $[u, V] \subset W$ (see Corollary 2.2(2c)), we see that W contains a codimension-one subspace of \mathfrak{t} , so W contains nontrivial hyperbolic elements. This contradicts the fact that $W \subset \mathfrak{m} + \text{unip } \mathfrak{p}_{\beta}$ does not contain nontrivial hyperbolic elements. \square

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